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DISSIPATIVE SYSTEMS

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1. Introduction

The research to be reported on here was completed recently by Hale, LaSalle, and Slemrod and a complete paper [1] on the subject is to appear. The research developed in the following manner. Billotti under the direction of LaSalle completed a study [2] in 1969 of dissipative retarded functional differential equations, and he and LaSalle then developed these results in a more general fashion [3] which, however, assumed the strong "smoothing" of initial data as occurs for retarded functional differential equations. Discussions of this work with Hale and Slemrod interested them in further generalizations. Slemrod saw how to do this for parabolic and hyperbolic partial differential equations and Hale for a wide class of functional differential equations of neutral type. It was then decided to unite the disparate points of view by identifying the general hypothesis which lead to the principal results, and the culmination of this collaborative effort is the subject of this report.

2. Processes

To present the background and the results succintly it is convenient

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to begin with the concept of a process (think of this as a "generalized nonautonomous dynamical system").

Let $R = (-\infty,\infty)$, $R^+ = [0,\infty)$, X be a Banach space and $u: R \times X \times R^+ \to X$. Define

$$(\sigma,t)x = u(\sigma,x,t), \quad (\sigma,t): \; X \to X.$$

X is the "state" space and interpret $(\sigma,t)x$ to be the state of the system at time $\sigma + t$ if initially the state at time σ was x.

The mapping u is said to define a <u>process</u> on X if u has the following properties:

(i) u is continuous.

(ii) $(\sigma,0) = I$, the identity mapping.

(iii) $(\sigma+s,t)(\sigma,s) = (\sigma,s+t)$, $\sigma \in R$ and $s,t \in R^+$.

Property (iii) corresponds to uniqueness in the forward direction of time. The (positive) <u>motion</u> or <u>orbit</u> through (σ,x) is $\bigcup_{t\geq 0} (\sigma,t)x$.

A <u>dynamical system</u> is an autonomous process: $(\sigma,t) = (0,t)$ for all $\sigma \in R$ and all $t \in R^+$. A motion is said to be <u>periodic</u> of period $\alpha > 0$ if $(\sigma,t+\alpha)x = (\sigma,t)x$ for all $t \in R^+$. A process is said to be <u>periodic</u> of period $\omega > 0$, if $(\sigma+\omega,t) = (\sigma,t)$ for all $\sigma \in R$ and all $t \in R^+$.

For a periodic process the Poincaré map $T: X \to X$ defined by

$$Tx = (\sigma,\omega)x$$

for some fixed ω defines a discrete dynamical system with a motion or orbit through x given by $\gamma^+(x) = \bigcup_{n=0}^{\infty} T^n x$. It follows easily that fixed points of T^k correspond to periodic motions of the periodic process of period $k\omega$. The limit set $L(x)$ of a discrete motion through x is

$$L(x) = \bigcap_{j=0}^{\infty} \text{Cl} \bigcup_{n=j}^{\infty} T^n x.$$

A set M in X is said to be postively invariant if $TM \subset M$, negatively invariant if $M \subset TM$ and invariant if $M = TM$. It is easy to see that

Lemma 1. If $\gamma^+(x)$ is precompact, then $L(x)$ is nonempty, compact and invariant.

Remark. It turns out later to be useful to note that the above result holds if x is replaced by an arbitrary compact set K.

In relation to applications one of the problems, as in the use of this lemma, in developing a general theory is to have results which depend upon determining boundedness of motions. One cannot, in general, give direct tests for compactness but can verify boundedness by use, for example, of Liapunov functions (see [1] for references to how this difficulty was overcome in developing a general stability theory).

3. The Principal Results to be Generalized

Let $f: R \times R^n \to R^n$ be continuous and define a system of ordinary differential equations

$$(1) \qquad \dot{x} = f(t,x).$$

Assume that the solution $\varphi(t,\sigma,\xi)$, $\varphi(\sigma,\sigma,\xi) = \xi$ for $\sigma \in R$ and $\xi \in R^n$ is unique, is defined for all $t \in R^n$, and depends continuously on (t,σ,ξ). Then $u(\sigma,\xi,t) = \varphi(\sigma+t,\sigma,\xi)$ defines a process on R^n.

For a periodic system of ordinary differential equations (1) ($f(t,x) = f(t+\omega,x)$ for some $\omega > 0$, all $t \in R$ and all $x \in R^n$) let T be the corresponding Poincaré map defined above.

Then (1) (or T) is said to be dissipative if there is a bounded set B in R^n such that given $x \in R^n$ there is an integer $n(x)$ with the property that $T^n x \in B$ for all $n \geq n(x)$. (It is sufficient to assume only that $T^{n(x)} \in B$.)

This concept of dissipativeness for $n = 2$ was first studied by Levinson [4] in 1949 and more general results can be found in [5], [6], and [7] The principal properties of dissipative systems, the objective of our generalization, are:

I. There is a maximal (nonempty) compact set J invariant under T.

II. J is globally asymptotically stable.

III. For some integer k_0, T^k has a fixed point for each $k \geq k_0$.

Levinson in [4] proved for $n = 2$, I and III (with $k_0 = 1$), and Pliss in [5] has the three results for general systems of ordinary differential equations. As will be pointed out later, it is now known for ordinary differential equations, that $k_0 = 1$ (there is always a periodic solution of period ω).

4. Retarded Functional Differential Equations

With $r \geq 0$ given let $C = C([-r,0],R^n)$ be the space of continuous functions mapping $[-r,0]$ into R^n with the topology of uniform convergence. For any continuous x defined on $[\sigma-r,\sigma+A)$, $A > 0$ and any $t \in [\sigma,\sigma+A)$ define x_t in C by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r,0]$. Let $f: R \times C \to R^n$ be continuous. A function $x = x(\sigma,\varphi)$ defined on and continuous on $[\sigma-r,\sigma+A)$ is said to be a solution of the retarded functional differential equation

$$\dot{x}(t) = f(t,x_t) \tag{2}$$

on $[\sigma,\sigma+A)$ with initial value φ at σ if $x_\sigma = \varphi$ and $x(t)$ satisfies (2) on $[\sigma,\sigma+A)$. Assume that each solution $x(\sigma,\varphi)$ exists and is unique on $[\sigma,\infty)$, and $x(\sigma,\varphi)t$ depends continuously on (σ,φ,t). Then $u(\sigma,\varphi,t) = x_{\sigma+t}(\sigma,\varphi)$ defines a process on C. Again assume (2) is periodic of period ω and T is the Poincaré map of C into itself defined by the periodic process u. Then the dissipative property is as before $(X = C)$.

H_1'') There is a bounded set B in X with the property that given $x \in X$ there is a positive integer $n(x)$ such that $T^n x \in B$ for all $n \geq n(x)$.

With the further, not unatural assumption, on f that f takes bounded sets of $R \times C$ into bounded sets of R^n, then the solutions of (2) smooth initial data. In fact, expressed in terms of T, retarded functional differential equations have the following smoothing property ($X = C$):

H_2'') There is an integer n_0 such that given a bounded set B in X there is a compact set B^* in X such that $T^n x \in B$ for $n = 0,1,2,\ldots,N (N \geq n_0)$ implies $T^n(x) \in B^*$ for $n = n_0, n_0 + 1, \ldots N$.

The integer n_0 is the length of time it takes to smooth the initial data. For the retarded functional differential equation (2) n_0 is the smallest integer such that $n_0\omega \geq r$ and for the ordinary differential equation (1) $n_0 = 0$ (since $X = R^n$ is locally compact). With H_2'') it is sufficient to assume in H_1'') only that $T^n(x) \in B$ for $n(x) \leq n \leq n(x) + n_0$; that is, only long enough to smooth.

6. Functional Differential Equations of Neutral Type

For a more general definition of functional differential equations of neutral type and basic theorems concerning solutions and their properties see [8] and [9]. Here we consider a more special case.

Let C, f, and x_t be as before. Consider, in addition, the continuous map $D: R \times C \to R^n$ of the form

$$D(t,\varphi) = \varphi(0) + B_1(t)\varphi(-r_1) + \cdots + B_k(t)\varphi(-r_k)$$

where $0 \leq r_j \leq r$ and the B_j are uniformly continuous and bounded for

$t \in R$. A function $x = x(\sigma,\varphi)$ defined and continuous on $[\sigma-r,\sigma+A)$, $A > 0$, is said to be a solution of the neutral functional differential equation

$$\frac{d}{dt} D(t,x_t) = f(t,x_t) \tag{3}$$

on $(\sigma,\sigma+A)$ with initial value φ at σ if $x_\sigma = \varphi$, $D(t,x_t)$ is continuously differentiable on $(\sigma,\sigma+A)$ and satisfies (3) on $(\sigma,\sigma+A)$. Assume for any $(\sigma,\varphi) \in R \times C$ that a solution of (3) exists on (σ,∞), is unique, and $x(\sigma,\varphi)(t)$ is continuous in (σ,φ,t). The function $u(\sigma,\varphi,t) = x_{\sigma+t}(\sigma,\varphi)$ defines a process on C.

In this generality one cannot expect solutions to be smoother than the initial data and further restrictions need to be placed on D (see [8]). More specifically, we shall assume that D is stable (see [8]); suffice it to say here that it is shown in [8] that D is stable if and only if the solutions of $D(t,x_t) = 0$ are uniformly asymptotically stable. If $D(t,\varphi)$ and $f(t,\varphi)$ are ω-periodic in t, D is stable, and f maps bounded sets of $R \times C$ into bounded sets of R^n, then from results in [8] and [10] it is not difficult to see that the Poincaré map T for this class of neutral functional differential equations has the following smoothing properties $(X = C)$:

H_2') To each bounded set B in X there corresponds a compact set B^* in X with the property that given $\varepsilon > 0$ there is an integer $n_0(\varepsilon,B)$ such that $T^n x \in B$ for $n \geq 0$ implies $T^n x \in B^*_\varepsilon$ for

$n \geq n_0(\varepsilon, B)$, where B^*_ε is an ε-neighborhood of B^*.

$H_4')$ For any compact set K in X, $\gamma^+(K) = \bigcup_{n=0}^{\infty} T^n K$ bounded implies $\gamma^+(K)$ is precompact.

The weaker smoothing here requires a stronger concept of dissipativeness:

$H_1')$ There is a bounded set B in X such that for each x in X there is a neighborhood O_x of x and an integer $N(x)$ such that $T^n O_x \subset B$ for $n \geq N(x)$.

7. Partial Differential Equations

Certain types of parabolic and hyperbolic partial differential equations are known to define processes on appropriate Sobolev spaces. In the hyperbolic case there is some smoothing of initial data but this is not so for hyperbolic equations. However, when it is known that a hyperbolic equation defines a process on two Sobolev spaces X and Y with $X \subset Y$ algebraically and topologically and with the injection map completely continuous, then the smoothing affect is replaced by the fact that a bounded orbit in X is compact in Y.

8. General Hypotheses

As indicated by the brief discussion of the situation for partial differential equations one will, in general, want to consider a transformation T of two spaces X and Y with X imbedded in Y as described above with the injection map assumed to be at least continuous. It then turns out that there are four hypotheses needed to obtain the generalizations of I, II, and III stated in Section 3. All of this requires considerable explanation and is more than we can enter into here (the reader is referred to [1]). The four hypotheses are of the following type:

H_1) A dissipative property

H_2) A smoothing property

H_3) A fixed point property

H_4) A smoothing property.

When $X = Y$ these hypotheses become

H_1') There is a bounded set $B \subset X$ such that for any $x \in X$, there is a neighborhood O_x of x and an integer $N(x)$ such that $T^n O_x \subset B$ for $n \geq N(x)$.

H_2') To each bounded set B in X there corresponds a compact set B^* in X with the property that given $\varepsilon > 0$ there is an integer $n_0(\varepsilon, B)$ such that $T^n x \in B$ for $n \geq 0$ implies $T^n x \in B^*_\varepsilon$ for $n \geq n_0(\varepsilon, B)$, where B^*_ε is an ε-neighborhood of B^*.

H_3') There is an integer k_0 such that for every closed bounded convex set $B \subset X$ and every integer $k \geq k_0$, if $T^n B$ is bounded for $0 \leq n \leq k$ and $T^k: B \to B$, then T^k has a fixed point in B.

H_4') For any compact set $B \subset X$, $\gamma^+(B)$ bounded implies $\gamma^+(B)$ precompact.

The results corresponding to I and II are implied by H_1), H_2), and H_4), and the fixed point property III follows from H_1), H_2) and H_3).

In the case of partial differential equations when the injection map is completely continuous it can be shown that H_1) implies H_1) - H_4), so that all that need be assumed is H_1). When there is smoothing, as in the case of retarded or neutral functional, differential equations, then H_1') implies H_1') - H_4') and again only a dissipative assumption is required.

Let us examine at least the case of retarded functional differential equations in some detail. Here we have the smoothing property H_2)" and need only assume the weaker form of dissipativeness H_1)". It can then be shown that H_1)" and H_2'') imply H_1') - H_4'). In fact, one obtains

<u>Theorem</u> 1. If T satisfies H_1'') and H_2''), then there is a compact set K in X with the property that given a compact set H in X there is an open neighborhood H_0 of H and an integer $N(H)$ such that $T^n(H_0) \subset K$ for all $n \geq N(H)$.

It can then be shown that $J = \bigcap_{n=0}^{\infty} T^n K$ is the maximal compact invariant. This is done by showing first of all that J is well-defined (does not depend on the choice of K from Theorem 1) and that $J = L(K)$. Being a limit set J is nonempty, compact and invariant (see the remark below Lemma 1) and it is easy to see that J is the maximal compact invariant set and is a global attraction. Proving that J is stable is more difficult. The fixed point property III follows readily from Theorem 1 and Schauder's fixed point theorem.

If one assumes, in addition to $H_1'')$ and $H_2'')$ that T maps bounded sets into bounded sets, then it follows that T^k is compact for $k \geq n_0$ and one can show using Browder's extension of the Schauder fixed point theorem that T^k has a fixed point for each $k \geq n_0$ (in III, $k_0 = n_0$). This result has also been given by Horn in [11] as a consequence of his extension of Schauder's fixed point theorem that is slightly different from Browder's. A similar result for retarded functional differential equations which are uniformly bounded and uniformly ultimately bounded was given by Yoshizawa in [12] (see also [13]). In addition, Yoshizawa assumes that the f in (2) satisfies a Lipschitz condition.

Thus for ordinary differential equations we know for dissipative ordinary differential equations there is always a periodic solution of period ω and for dissipative retarded functional differential equations when the solution map maps bounded sets into bounded sets and $\omega \geq r$ there is a periodic solution of period ω.

One suspects that there should be better results and from conversations at this meeting with G. Stephen Jones and J. Hale, it seems clear that a dissipative retarded functional differential equation always has a periodic solution of period ω without any further assumptions.

From subsequent conversations with J. Hale it seems that a similar result is true for a restricted class of neutral functional differential equations (it appears necessary, for example, to assume that the operator D is autonomous).

9. Concluding Remarks

The rather abstract theory presented here shows how the theory of dissipative systems of ordinary differential equations can be extended to include a wide class of functional and partial differential equations. Since the basic hypotheses are all in terms of boundedness, finding sufficient conditions in terms of Liapunov functions is not too difficult and we are undertaking now to work out some nontrivial examples to illustrate how the theory can be applied.

References

[1] J.K. Hale, J.P. LaSalle, and Marshall Slemrod, Theory of a general class of dissipative process, to appear in J. Math. Anal. and Appl.

[2] J.E. Billotti, Dissipative Functional Differential Equations, Ph.D. dissertation, Brown University, 1969.

[3] J.E. Billotti and J.P. LaSalle, Periodic dissipative processes, to appear in the Bull. Amer. Math. Soc.

[4] N. Levinson, Transformation theory of non-linear differential equations of second order, Annals of Math., 45(1944), 723-737.

[5] V.A. Pliss, Nonlocal Problems of the Theory of Oscillations, Academic Press, New York, 1966 (translation of 1964 Russian edition).

[6] J.P. LaSalle, A study of synchronous asymptotic stability, Annals of Math., 65(1957), 571-581.

[7] R. Reissig, G. Sansone, and R. Conti, Nonlineare Differentialgleichungen Höherer, Ordnung, Edizioni Cremonese, Roma, 1969.

[8] A.M. Cruz and J.K. Hale, Stability of functional differential equations of neutral type, J. Diff. Eqns., 7(1970), 334-355.

[9] J.K. Hale and A.M. Cruz, Existence and continuous dependence for hereditary systems, Annali Mat. Pura Appl. (4)(85)(1970), 63-81.

[10] J.K. Hale, A class of neutral equations with the fixed point property, Proc. Nat. Acad. Sci. U.S.A., 67(1970), 136-137.

[11] W.A. Horn, Some fixed point theorems for compact maps and flows in Banach spaces, Trans. Amer. Math. Soc., 149(1970), 391-404.

[12] Taro Yoshizawa, Ultimate boundedness of solutions and periodic solution of functional differential equations, Colloques Internationaux sur les Vibrations Forcées dans les Systèmes Non-linéaires, Marseille, September 1964, 167-179.

[13] Taro Yoshizawa, Stability Theory by Liapunov's Second Method, Math. Soc. of Japan Publication, Tokyo, 1966.